

PERIODIC ALGEBRAS GENERATED BY GROUPS

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ABSTRACT. We consider algebras with basis numerated by elements of a group G . We fix a function f from $G \times G$ to a ground field and give a multiplication of the algebra which depends on f . We study the basic properties of such algebras. In particular, we find a condition on f under which the corresponding algebra is a Leibniz algebra. Moreover, for a given subgroup \widehat{G} of G we define a \widehat{G} -periodic algebra, which corresponds to a \widehat{G} -periodic function f , we establish a criterion for the right nilpotency of a \widehat{G} -periodic algebra. In addition, for $G = \mathbb{Z}$ we describe all $2\mathbb{Z}$ - and $3\mathbb{Z}$ -periodic algebras. Some properties of $n\mathbb{Z}$ -periodic algebras are obtained.

1. INTRODUCTION

Infinite dimensional algebras were introduced in mathematics at the beginning of the last century, they have had a considerable development during the past 40 years, from affine Lie algebras and loop groups to quantum groups in their various flavors. They also have found an ever increasing variety of applications in many domains of physics, from various aspects of solid state physics to most sophisticated models of quantum field theory, see, e.g., [2], [3], [8], [16].

In the survey article [18] the author discusses some old and some new open questions on infinite-dimensional algebras. The paper describes interactions between combinatorial group theory, Lie algebras and infinite-dimensional associative algebras. In [5] a tame filtration of an algebra is defined by the growth of its terms, which has to be majorated by an exponential function. The notion of tame filtration is useful in the study of possible distortion of degrees of elements when one algebra is embedded as a subalgebra in another algebra. These authors consider the case of associative or Lie algebras in the case of tame filtration of an algebra can be induced from the degree filtration of a larger algebra.

Whereas that the theory of finite dimensional algebras is well developed in a systematic way, it is fair to say that this is not get the case for the theory of infinite dimensional algebras (see, however, e.g., [5], [7], [8], [18] and references there in). In this paper we consider algebras over a field K , with basis set $\{e_a, a \in G\}$, which is indexed by elements of a group (G, \circ) . The multiplication table is given as $e_a e_b = f(a, b) e_{a \circ b}$, where t is a fixed element of G and f a map of a Cartesian square of G into the field K . A construction of this type for an n -ary algebra was first considered in [11]. For an infinite group G our construction gives an infinite dimensional algebra.

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For a given subgroup \widehat{G} of G we define a notion of \widehat{G} -periodic algebra and study some its basic properties. Let us point out that corresponding notions of \widehat{G} -periodic Gibbs measures and periodic p -harmonic functions were considered in [13], [14], [15], respectively. Whereas, the notion of \widehat{G} -periodic algebra can be considered in arbitrary variety of algebras.

In the present work we limit ourselves to the study a particular case of \widehat{G} -periodic algebras. Namely, we study \widehat{G} -periodic Leibniz algebras with additive group G .

It is known that Leibniz algebras are generalization of Lie algebras [9]. A lot of papers has been devoted to the description of finite dimensional Leibniz algebras. In the study of a variety of algebras the classification of algebras in low dimensions plays a crucial role. Moreover, some conjectures can be verified in low dimensions. In the past much work has been invested into the classification of various varieties of algebras over the field of the complex numbers and fields of positive characteristics. We recall that the description of finite dimensional complex Lie algebras has been reduced to the classification of nilpotent Lie algebras, which have been completely classified up to dimension 7 (see [10], [17]). In the case of Leibniz algebras the problem of classification of complex Leibniz algebras has been solved up to dimension 4 [1], [4]. However, the classification of infinite dimensional algebras is more complicated. In this paper we give a construction of certain infinite dimensional algebras which are relatively simple to describe.

In our case from the Leibniz identity, we derive the functional equation for f . Thus, the problem of the classification of corresponding Leibniz algebras can be reduced to the problem of the description of the functions f up to a non-degenerate basis transformation. Moreover in periodic cases our construction reduces the study of infinite dimensional algebras to the study of finite dimensional matrices.

2. PRELIMINARIES

Let A be an arbitrary algebra and let $\{e_1, e_2, \dots\}$ be a basis of the algebra A . The table of multiplication on the algebra is defined by the products of the basic elements, namely, $e_i e_j = \sum_k \gamma_{i,j}^k e_k$, where $\gamma_{i,j}^k$ are the structural constants.

We recall that Leibniz algebras are defined by the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

If the identity $[x, x] = 0$ holds in Leibniz algebras then the Leibniz identity coincides with the Jacobi identity. Thus, the Leibniz algebras are a “non commutative” analogue of the Lie algebras.

Let L be a Leibniz algebra, we define the lower central sequence and, respectively, the derived sequence by:

$$\begin{aligned} L^1 &= L, & L^{n+1} &= [L^n, L], & n &\geq 1, \\ \text{resp. } L^{[0]} &= L, & L^{[k+1]} &= [L^{[k]}, L^{[k]}], & k &\geq 1. \end{aligned}$$

Definition 2.1. *An algebra L is called nilpotent (solvable) if there exists some $s \in \mathbb{N}$ (respectively, $t \in \mathbb{N}$) such that $L^s = 0$ (respectively, $L^{[t]} = 0$). The minimal such number s (resp. t) is called index of nilpotency (resp. solvability).*

For any element x of L we define the operator R_x of right multiplication as follows

$$R_x : z \rightarrow [z, x], \quad z \in L$$

Definition 2.2. Let G be an additive group. An algebra A is called G -graded algebra if A can be decomposed into the direct sum of vector spaces (i.e. $A = \sum_{g \in G} \oplus A_g$) and $A_g A_h \subset A_{g+h}$, $\forall g, h \in G$.

3. AN ALGEBRA GENERATED BY A GROUP

Let K be a field. Fix an element t of a group (G, \circ) and consider the algebra $A_G(f, t) = \langle e_a | a \in G \rangle$ given by the following multiplication:

$$e_a e_b = f(a, b) e_{a \circ b}, \quad (3.1)$$

where $f : G \times G \rightarrow K$ is a given function.

Denote by $\mathbf{1}$ the unit element of the group G . Note that the set $\{e_a : a \in G\}$ is a group with binary operation $e_a * e_b = e_{a \circ b}$.

Proposition 3.1. Take $t \in \bigcap_{\substack{H \leq G: \\ H \neq \{1\}}} H$.

- 1) If H and J are subgroups of the group G and f is such that $f(h, t^{-1}) \neq 0$, for any $h \in H$. Then $A_H(f, t)$ is a subalgebra of $A_J(f, t)$ iff H is a subgroup of J .
- 2) If $M \subset G$ with $M \circ M \circ t \subseteq M$ then $A_M(f, t)$ is a subalgebra $A_G(f, t)$, but the opposite is not true, in general.
- 3) Let H be a subgroup of G and f is such that $f(h, g) \neq 0$ for all $h \in H, g \in G$. Then $A_H(f, t)$ is an ideal of $A_G(f, t)$ iff $H = G$.

Proof. 1) If $H \leq J$ then $\{e_h | h \in H\} \subset \{e_j | j \in J\}$, consequently $A_H(f, t)$ is a subalgebra of $A_J(f, t)$. Now assume $A_H(f, t)$ is a subalgebra of $A_J(f, t)$ then for any $h \in H$ we consider

$$e_h e_{t^{-1}} = f(h, t^{-1}) e_h.$$

Since $f(h, t^{-1}) \neq 0$, we have $e_h \in \{e_j | j \in J\}$. Hence $h \in J$.

2) Straightforward.

3) If $H = G$ then $A_H(f, t) = A_G(f, t)$. If $A_H(f, t)$ is an ideal of $A_G(f, t)$, then for any $e_h \in A_H(f, t)$ and $e_g \in A_G(f, t)$ we have

$$e_h e_g = f(h, g) e_{h \circ g \circ t} \in A_H(f, t).$$

This gives $h \circ g \circ t \in H$, which is equivalent to $g \in H$, for all $g \in G$. □

Proposition 3.2. Let G be a commutative group. Then the algebra $A_G(f, t)$ is solvable of index m if and only if there exists $m \in \mathbb{N}$ such that

$$\prod_{k=0}^{m-1} \prod_{s=0}^{2^{m-k-1}-1} f \left(t^{2^k-1} \circ \prod_{q=1}^{2^k} \circ a_{2^{k+1}s+q}, \quad t^{2^k-1} \circ \prod_{l=1}^{2^k} \circ a_{2^{k+1}s+2^k+l} \right) = 0, \quad (3.2)$$

for any $a_1, \dots, a_{2^m} \in G$.

Proof. First we shall prove the following formula

$$\begin{aligned} & (\dots (e_{a_1} e_{a_2}) \dots (e_{a_{2r-1}} e_{a_{2r}}) \dots) = \\ & \prod_{k=0}^{r-1} \prod_{s=0}^{2^{r-k-1}-1} f \left(t^{2^k-1} \circ \prod_{q=1}^{2^k} \circ a_{2^{k+1}s+q}, \quad t^{2^k-1} \circ \prod_{l=1}^{2^k} \circ a_{2^{k+1}s+2^k+l} \right) e_{t^{2^r-1} \circ \prod_{i=1}^{2^r} \circ a_i}. \end{aligned} \quad (3.3)$$

We use mathematical induction by r . For $r = 1$ and $r = 2$ using the equality (3.1) we obtain

$$\begin{aligned} r = 1 : \quad & e_{a_1} e_{a_2} = f(a_1, a_2) e_{a_1 \circ a_2 \circ t}. \\ r = 2 : \quad & (e_{a_1} e_{a_2})(e_{a_3} e_{a_4}) = f(a_1, a_2) e_{a_1 \circ a_2 \circ t} f(a_3, a_4) e_{a_3 \circ a_4 \circ t} = \\ & f(a_1, a_2) f(a_3, a_4) f(t \circ a_1 \circ a_2, t \circ a_3 \circ a_4) e_{a_1 \circ a_2 \circ a_3 \circ a_4 \circ t^3}. \end{aligned}$$

Thus the equality (3.3) is true for $r = 1$ and $r = 2$. Assume that (3.3) is true for r and prove it for $r + 1$. By the assumption of the induction we get

$$(\dots (e_{a_1} e_{a_2}) \dots (e_{a_{2r-1}} e_{a_{2r}}) \dots) (\dots (e_{a_{2r+1}} e_{a_{2r+2}}) \dots (e_{a_{2r+1-1}} e_{a_{2r+1}}) \dots) = U \times V \times W,$$

where

$$\begin{aligned} U &= \prod_{k=0}^{r-1} \prod_{s=0}^{2^{r-k-1}-1} f \left(t^{2^k-1} \circ \prod_{q=1}^{2^k} \circ a_{2^{k+1}s+q}, \quad t^{2^k-1} \circ \prod_{l=1}^{2^k} \circ a_{2^{k+1}s+2^k+l} \right), \\ V &= \prod_{k=0}^{r-1} \prod_{s=0}^{2^{r-k-1}-1} f \left(t^{2^k-1} \circ \prod_{q=1}^{2^k} \circ a_{2^r+2^{k+1}s+q}, \quad t^{2^k-1} \circ \prod_{l=1}^{2^k} \circ a_{2^r+2^{k+1}s+2^k+l} \right), \\ W &= e_{t^{2^r-1} \circ \prod_{i=1}^{2^r} \circ a_i} e_{t^{2^r-1} \circ \prod_{i=1}^{2^r} \circ a_{2^r+i}} = \\ & f \left(t^{2^r-1} \circ \prod_{i=1}^{2^r} \circ a_i, \quad t^{2^r-1} \circ \prod_{i=1}^{2^r} \circ a_{2^r+i} \right) e_{t^{2^r+1-1} \circ \prod_{i=1}^{2^r+1} \circ a_i}. \end{aligned}$$

In V we change s with $s = s' - 2^{r-k-1}$ then

$$V = \prod_{k=0}^{r-1} \prod_{s'=2^{r-k-1}}^{2^{r-k}-1} f \left(t^{2^k-1} \circ \prod_{q=1}^{2^k} \circ a_{2^{k+1}s'+q}, \quad t^{2^k-1} \circ \prod_{l=1}^{2^k} \circ a_{2^{k+1}s'+2^k+l} \right).$$

Consequently, we get

$$U \times V = \prod_{k=0}^{r-1} \prod_{s=0}^{2^{r-k}-1} f \left(t^{2^k-1} \circ \prod_{q=1}^{2^k} \circ a_{2^{k+1}s+q}, \quad t^{2^k-1} \circ \prod_{l=1}^{2^k} \circ a_{2^{k+1}s+2^k+l} \right).$$

Hence we have

$$U \times V \times W = \prod_{k=0}^r \prod_{s=0}^{2^{r-k}-1} f \left(t^{2^k-1} \circ \prod_{q=1}^{2^k} \circ a_{2^{k+1}s+q}, \quad t^{2^k-1} \circ \prod_{l=1}^{2^k} \circ a_{2^{k+1}s+2^k+l} \right) e_{t^{2^r+1-1} \circ \prod_{i=1}^{2^r+1} \circ a_i}.$$

This gives (3.3) for $r + 1$. By formula (3.3) one can see that the algebra $A_G(f, t)$ is solvable of index m if and only if the condition (3.2) is satisfied. \square

Denote by A_g the vector space Ke_g . Then we have $A_G(f, 0) = \sum_{g \in G} \oplus A_g$. Since $A_g A_h = \{\alpha e_g \beta e_h = \alpha \beta f(g, h) e_{g+h} \mid \alpha, \beta \in K, g, h \in G\}$ we get $A_g A_h \subseteq A_{g+h}$. Hence the algebra $A_G(f, 0)$ is a graded algebra.

Let G be an additive group. We shall find a condition on f under which the algebra $A_G(f, t)$ will be a Leibniz algebra. From the Leibniz identity we get for the function f the following equation

$$f(b, c)f(a, b + c + t) = f(a, b)f(a + b + t, c) - f(a, c)f(a + c + t, b). \quad (3.4)$$

For a given $t \in G$ we set

$$F_t = \{f : \text{for the given } t, f \text{ is a solution of (3.4)}\}.$$

Denote by $L(f, t)$ the algebra which is given by $f \in F_t$.

Proposition 3.3. *For any $t, t' \in G$ and $f \in F_t$ there exists $g \in F_{t'}$ such that $L(f, t) \cong L(g, t')$ (where \cong means algebraically isomorph).*

Proof. Take the isomorphism $\varphi(e_i) = e'_{i+t-t'}$. Then

$$\begin{aligned} e'_a e'_b &= e_{a-(t-t')} e_{b-(t-t')} = f(a - (t - t'), b - (t - t')) e_{a+b-2(t-t')+t} = \\ &= f(a - (t - t'), b - (t - t')) e'_{a+b+t'}. \end{aligned}$$

For a given t' we define $g(a, b) := f(a - (t - t'), b - (t - t'))$. Now we shall check the identity (3.4) for g . For the elements $a' = a + t' - t$, $b' = b + t' - t$, $c' = c + t' - t$ we have

$$f(b', c')f(a', b' + c' + t) = f(a', b')f(a' + b' + t, c') - f(a', c')f(a' + c' + t, b'),$$

consequently

$$g(b, c)g(a, b + c + t') = g(a, b)g(a + b + t', c) - g(a, c)g(a + c + t', b).$$

Hence, $g \in F_{t'}$ and $L(f, t) \cong L(g, t')$. \square

Let us present an example:

Example 1. *Consider the group $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, then function f is defined on $\mathbb{Z}_2 \times \mathbb{Z}_2$. The four values of function f can be represented in the form of 2×2 matrix, i.e. by matrix $(f(a, b))_{a, b = \bar{0}, \bar{1}}$. It easy to see that*

$$\begin{aligned} F_{\bar{0}} &= \{(f(a, b))_{a, b = \bar{0}, \bar{1}} : f \text{ is a solution of (3.4) for } t = \bar{0}\} = \\ &= \left\{ \begin{pmatrix} 0 & 0 \\ \alpha_{10} & 0 \end{pmatrix}; \begin{pmatrix} 0 & -\alpha_{10} \\ \alpha_{10} & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 0 & \alpha_{11} \end{pmatrix}, \text{ where } \alpha_{10}, \alpha_{11} \in K \right\}, \\ F_{\bar{1}} &= \{(f(a, b))_{a, b = \bar{0}, \bar{1}} : f \text{ is a solution of (3.4) for } t = \bar{1}\} = \\ &= \left\{ \begin{pmatrix} 0 & \alpha_{01} \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & -\alpha_{10} \\ \alpha_{10} & 0 \end{pmatrix}; \begin{pmatrix} \alpha_{00} & 0 \\ 0 & 0 \end{pmatrix}, \text{ where } \alpha_{00}, \alpha_{01}, \alpha_{10} \in K \right\}. \end{aligned}$$

Consequently, the construction gives 6 Leibniz algebras $L_{\bar{i},j}(\theta, t)$, $i = 0, 1$, $j = 1, 2, 3$, $\theta = \alpha_{00}, \alpha_{01}, \alpha_{11}$, $t = \bar{0}, \bar{1}$. Moreover, one can check that $L_{\bar{0},i}(\theta, t) \cong L_{\bar{1},i}(\theta, t')$.

4. PERIODIC ALGEBRAS

Let \widehat{G} be a subgroup of an additive group G , then the group G is decomposable into cosets with respect to this subgroup: $G/\widehat{G} = \{g_0 + \widehat{G}, g_1 + \widehat{G}, \dots, g_{n-1} + \widehat{G}\}$, where n is index of the subgroup \widehat{G} in G .

Let \widehat{G} be a subgroup of index n for G . Then $G/\widehat{G} = \{G_0, \dots, G_{n-1}\}$, where $G_0 = \widehat{G}$.

Definition 4.1. The function $f : G \times G \rightarrow K$ is called \widehat{G} -periodic, if $f(a, b) = \alpha_{ij}$ for all $a \in G_i$, $b \in G_j$.

In other words the function f is periodic if its value at the point $(a, b) \in G \times G$ does not depend on a and b , but the value depends only on the cosets to which a and b belong.

Definition 4.2. The algebra $A_G(f, t)$ is called \widehat{G} -periodic, if it corresponds to a \widehat{G} -periodic function f .

Let \widehat{G} be a subgroup of an additive group G and $A_{\widehat{G}}(f, 0)$ be a \widehat{G} -periodic algebra. For the factor group $G/\widehat{G} = \{\widehat{G}, g_1 + \widehat{G}, \dots\}$ we set $A_{g_i} = \{\alpha e_{g_i+h} : h \in \widehat{G}, \alpha \in K\}$. Then with respect to these sets $A_{\widehat{G}}(f, 0)$ is a graded algebra.

If $a \in G_i$, $b \in G_j$ then instate $f(a, b) = \alpha_{ij}$ we write $f(a, b) = \alpha_{\bar{a}, \bar{b}}$ i.e. $\alpha_{ij} = \alpha_{\bar{a}, \bar{b}}$. In other words \bar{a} denotes the number of the coset where belongs a .

Theorem 4.3. Let \widehat{G} be a subgroup of index $n \geq 1$, a \widehat{G} -periodic algebra $A_G(f, t)$ is right nilpotent if and only if

$$\alpha_{\bar{a}_1, \bar{a}_2} \alpha_{\overline{a_1+a_2}, \bar{a}_3} \dots \alpha_{\overline{a_1+a_2+\dots+a_{k-1}}, \bar{a}_k} = 0,$$

for any k , $k \leq n$ and for arbitrary $a_1, \dots, a_k \in G$ with $a_2 + \dots + a_k \in \widehat{G}$.

Proof. It is known that the algebra A is right nilpotent iff there exists k such that $R_{x_2} R_{x_3} \dots R_{x_k} = 0$ for arbitrary $x_2, \dots, x_k \in A$. It is enough to check this condition for $x_2 = e_{a_2}, \dots, x_k = e_{a_k}$.

Necessity. Assume A is right nilpotent. We have

$$R_{e_{a_k}} R_{e_{a_{k-1}}} \dots R_{e_{a_2}}(e_{a_1}) = \alpha_{\bar{a}_1, \bar{a}_2} \alpha_{\overline{a_1+a_2}, \bar{a}_3} \dots \alpha_{\overline{a_1+a_2+\dots+a_{k-1}}, \bar{a}_k} e_{a_1+a_2+\dots+a_k+(k-1)t} = 0. \quad (4.1)$$

If $\overline{a_1 + a_2 + \dots + a_k} = \bar{a}_1$ and

$$\alpha_{\bar{a}_1, \bar{a}_2} \alpha_{\overline{a_1+a_2}, \bar{a}_3} \dots \alpha_{\overline{a_1+a_2+\dots+a_{k-1}}, \bar{a}_k} \neq 0$$

then we can consider $(R_{e_{a_k}} R_{e_{a_{k-1}}} \dots R_{e_{a_2}}(e_{a_1}))^m$ which is not zero for any $m \geq 1$. So the condition of the theorem is necessary.

Sufficiency. Assume

$$\alpha_{\bar{a}_1, \bar{a}_2} \alpha_{\overline{a_1+a_2}, \bar{a}_3} \dots \alpha_{\overline{a_1+a_2+\dots+a_{k-1}}, \bar{a}_k} = 0,$$

for any k , $k \leq n$ and for arbitrary $a_1, \dots, a_k \in G$ with $a_2 + \dots + a_k \in \widehat{G}$. We shall prove that $A_G(f, t)$ is right nilpotent. Take $k > n$, then at least two of the following numbers coincide:

$$\overline{a_1 + a_2}, \overline{a_1 + a_2 + a_3}, \dots, \overline{a_1 + \dots + a_{k+1}}.$$

Let $\overline{a_1 + \dots + a_p} = \overline{a_1 + \dots + a_q}$, $1 \leq p - q \leq n$, i.e. $a_{p+1} + \dots + a_q \in \widehat{G}$. From this condition we have

$$\alpha_{\overline{a_1 + \dots + a_p}, \overline{a_{p+1}}} \alpha_{\overline{a_1 + \dots + a_{p+1}}, \overline{a_{p+2}}} \dots \alpha_{\overline{a_1 + a_2 + \dots + a_{q-1}}, \overline{a_q}} = 0.$$

This implies that $R_{e_{a_k}} R_{e_{a_{k-1}}} \dots R_{e_{a_2}}(e_{a_1}) = 0$ for any $k > n$. \square

Proposition 4.4. *Assume H, J are subgroups of G with finite indexes such that $|G : J|$ divides $|G : H|$ (where $| \cdot |$ stands for order). Then any J -periodic algebra is H -periodic, but there are H -periodic algebras which are not J -periodic.*

Proof. Any J -periodic algebra is represented by an $n \times n$ matrix $A_n = (\alpha_{ij})_{i,j=0,\dots,n-1}$, where $n = |G : J|$. Any H -periodic algebra is given by an $nm \times nm$ matrix $B_{nm} = (\beta_{ij})_{i,j=0,\dots,nm-1}$, where $nm = |G : H|$. It is easy to see that a given J -periodic algebra is H -periodic with $\beta_{nk+i,nl+j} = \alpha_{i,j}$, $i, j = 0, \dots, n-1$, $k, l = 0, \dots, m-1$. Consider an H -periodic algebra with $\beta_{n,n+1} \neq \alpha_{0,1}$, then this algebra is not J -periodic. \square

Denote

$$F_{t, \widehat{G}}^{per} = \{f : f \text{ is } \widehat{G} - \text{periodic and satisfies (3.4)}\}.$$

Proposition 4.5. *For any $t \in \widehat{G}$, $f \in F_{t, \widehat{G}}^{per}$ and $t' \in G_i$, $1 \leq i \leq n-1$ there exists $g \in F_{t', \widehat{G}}^{per}$ such that $L(f, t) \cong L(g, t')$.*

Proof. Note that $t - t' \in G_i$. Since f is periodic, we have $f(a - (t - t'), b - (t - t')) = \lambda_{jk}$ for $a \in t' + G_j$, $b \in t' + G_k$. We define $g(a, b) = f(a - (t - t'), b - (t - t')) = \lambda_{jk}$. It is easy to see that the function g satisfies equation (3.4). The periodicity of g follows from the periodicity of f . \square

Remark 4.6. *Using $f(a + t, b + t) = f(a, b)$ for all $t \in \widehat{G}$ one can show that for any $f \in F_{0, \widehat{G}}^{per}$ and $t \in \widehat{G}$ the \widehat{G} -periodic Leibniz algebra $L(f, 0)$ is isomorphic to the \widehat{G} -periodic algebra $L(f, t)$.*

Now we shall study the isomorphic character of the following algebras

$$L(f, t) : e_a e_b = f(a, b) e_{a+b+t} \text{ and } L(g, t) : e_a e_b = g(a, b) e_{a+b+t}.$$

Let φ is an isomorphism and it is given by a matrix (γ_{cd}) , i.e. $\varphi(e_c) = \sum_{d \in G} \gamma_{cd} e_d$. Then

$$\varphi(e_a) \varphi(e_b) = \left(\sum_{d \in G} \gamma_{ad} e_d \right) \left(\sum_{l \in G} \gamma_{bl} e_l \right) = \sum_{d, l \in G} \gamma_{ad} \gamma_{bl} f(d, l) e_{d+l+t}.$$

On the other hand, we have

$$\varphi(e_a)\varphi(e_b) = g(a, b)\varphi(e_{a+b+t}) = g(a, b) \sum_{m \in G} \gamma_{a+b+t, m} e_m.$$

Comparing the coefficients of the basis elements for any m we get

$$g(a, b)\gamma_{a+b+t, m} = \sum_{d \in G} \gamma_{a, d}\gamma_{b, m-d-t}f(d, m-d-t). \quad (4.2)$$

In the periodic case we reduced the problem to the case $t = 0$. Moreover, in this case the group and some subgroup of it are given. Thus for any k the equality (4.2) with $f(i, j) = \alpha_{ij}$, $g(i, j) = \beta_{ij}$ has the following form:

$$\beta_{i,j}\gamma_{i+j,k} = \sum_{s \in \mathbb{Z}} \gamma_{i,s}\gamma_{j,k-s}\alpha_{\overline{s}, \overline{k-s}}. \quad (4.3)$$

Assume $f(a, b) = \alpha_{i,j}$ for $a \in g_i + \widehat{G}$, $b \in g_j + \widehat{G}$.

If $t \in g_s + \widehat{G}$, then by (3.4) we get:

- 1) If $a, b, c \in g_i + \widehat{G}$ then $\alpha_{ii}\alpha_{i, \overline{2i+s}} = 0$.
- 2) If $a, b \in g_i + \widehat{G}$, $c \in g_j + \widehat{G}$ then $\alpha_{ij}\alpha_{i, \overline{i+j+s}} = \alpha_{ii}\alpha_{\overline{2i+s}, j} - \alpha_{ij}\alpha_{\overline{i+j+s}, i}$.
- 3) If $a, c \in g_i + \widehat{G}$, $b \in g_j + \widehat{G}$ then $\alpha_{ji}\alpha_{i, \overline{i+j+s}} = \alpha_{ij}\alpha_{\overline{i+j+s}, i} - \alpha_{ii}\alpha_{\overline{2i+s}, j}$.
- 4) If $b, c \in g_i + \widehat{G}$, $a \in g_j + \widehat{G}$ then $\alpha_{ii}\alpha_{j, \overline{2i+s}} = 0$.
- 5) If $a \in g_i + \widehat{G}$, $b \in g_j + \widehat{G}$, $c \in g_k + \widehat{G}$ then $\alpha_{jk}\alpha_{i, \overline{j+k+s}} = \alpha_{ij}\alpha_{\overline{i+j+s}, k} - \alpha_{ik}\alpha_{\overline{i+k+s}, j}$.

Proposition 4.7. Assume H, J are subgroups of G with finite indexes such that $|G : J|$ divides $|G : H|$. Let $G/J = \{G_0, \dots, G_{n-1}\}$ be the factor group with $G_i + G_j = G_{i+j(\text{mod } n)}$, $G/H = \{H_0, \dots, H_{nm-1}\}$ be the factor group with $H_i + H_j = H_{i+j(\text{mod } nm)}$. Then any J -periodic Leibniz algebra L is H -periodic.

Proof. Let $L(f, t)$ be a J -periodic Leibniz algebra which corresponds to the matrix $A_n = (\alpha_{ij})_{i,j=0, \dots, n-1}$. Using notions of the proof of Proposition 4.4 and also taking $\beta_{nk+i, nl+j} = \alpha_{i,j}$, $i, j = 0, \dots, n-1$; $k, l = 0, \dots, m-1$, we have that the corresponding algebra $L(f, t)$ is H -periodic. It remains to show that $L(f, t)$ satisfies the Leibniz identity with respect to the β_{ij} which were defined above. Let $G/H = \{H_0, \dots, H_{nm-1}\}$ be the corresponding factor group. Taking $a \in H_i$, $b \in H_j$, $c \in H_p$ and $t \in H_q$ then using the condition of the proposition, from 1)-5) mentioned above we get

$$\beta_{j,p}\beta_{i,j+p+q(\text{mod } nm)} = \beta_{i,j}\beta_{i+j+q(\text{mod } nm), p} - \beta_{i,p}\beta_{i+p+q(\text{mod } nm), j}. \quad (4.4)$$

Assume $i = nn_1 + i'$, $j = nn_2 + j'$, $p = nn_3 + p'$, $q = nn_4 + q'$, with $i', j', p', q' \in \{0, \dots, n-1\}$ then (4.4) becomes:

$$\alpha_{j', p'}\alpha_{i', j'+p'+q'(\text{mod } n)} = \alpha_{i', j'}\alpha_{i'+j'+q'(\text{mod } n), p'} - \alpha_{i', p'}\alpha_{i'+p'+q'(\text{mod } n), j'}, \quad (4.5)$$

which holds since $L(f, t)$ is a Leibniz algebra with respect to α_{ij} . \square

Let us give some examples for our construction:

4.8. **$2\mathbb{Z}$ -periodic Leibniz algebras.** Consider the case $G = \mathbb{Z}$, $\widehat{G} = 2\mathbb{Z}$.

In this case $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$.

Let $t \in \bar{0}$. We consider all possible cases for $a, b, c \in \{\bar{0}, \bar{1}\}$:

- 1) $a, b, c \in \bar{0} \Rightarrow \alpha_{00} = 0$.
- 2) $a, b \in \bar{0}, c \in \bar{1} \Rightarrow \alpha_{01}^2 = \alpha_{00}\alpha_{01} - \alpha_{01}\alpha_{10}$.
- 3) $a, c \in \bar{0}, b \in \bar{1} \Rightarrow \alpha_{10}\alpha_{01} = \alpha_{01}\alpha_{10} - \alpha_{00}\alpha_{01}$.
- 4) $b, c \in \bar{0}, a \in \bar{1} \Rightarrow \alpha_{00}\alpha_{10} = 0$.
- 5) $a, b, c \in \bar{1} \Rightarrow \alpha_{11}\alpha_{10} = 0$.
- 6) $a, b \in \bar{1}, c \in \bar{0} \Rightarrow \alpha_{10}\alpha_{11} = \alpha_{11}\alpha_{00} - \alpha_{10}\alpha_{11}$.
- 7) $a, c \in \bar{1}, b \in \bar{0} \Rightarrow \alpha_{01}\alpha_{11} = \alpha_{10}\alpha_{11} - \alpha_{11}\alpha_{00}$.
- 8) $b, c \in \bar{1}, a \in \bar{0} \Rightarrow \alpha_{11}\alpha_{00} = 0$.

After simplifications we get

$$\begin{cases} \alpha_{00} = 0 \\ \alpha_{01}(\alpha_{01} + \alpha_{10}) = 0 \\ \alpha_{11}\alpha_{10} = 0 \\ \alpha_{01}\alpha_{11} = 0. \end{cases}$$

Consider the all possible cases:

Case 1. $\alpha_{11} = 0$. Then $\alpha_{01}(\alpha_{01} + \alpha_{10}) = 0$.

Case 1.1. $\alpha_{01} = 0$. Then α_{10} is an arbitrary parameter.

Case 1.2. $\alpha_{01} \neq 0$. Then $\alpha_{01} = -\alpha_{10}$.

Case 2. $\alpha_{11} \neq 0$. Then $\alpha_{01} = \alpha_{10} = 0$.

Thus for $t \in \bar{0}$ we obtain the following matrices for the structural constants:

$$\begin{pmatrix} 0 & 0 \\ \alpha_{10} & 0 \end{pmatrix}; \begin{pmatrix} 0 & -\alpha_{10} \neq 0 \\ \alpha_{10} & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 0 & \alpha_{11} \neq 0 \end{pmatrix}.$$

By scaling the basis we obtain the following three $2\mathbb{Z}$ -periodic Leibniz algebras:

$$L_{\bar{0}1}(\alpha, t) : e_{2k-1}e_{2m} = \alpha e_{2(k+m)-1+t}, \quad \alpha \in \{0, 1\};$$

$$L_{\bar{0}2}(t) : \begin{cases} e_{2k}e_{2m-1} = -e_{2(k+m)-1+t}, \\ e_{2k-1}e_{2m} = e_{2(k+m)-1+t}, \end{cases};$$

$$L_{\bar{0}3}(t) : e_{2k-1}e_{2m-1} = e_{2(k+m-1)+t}.$$

Similarly for $t \in \bar{1}$ we obtain the following periodic Leibniz algebras:

$$L_{\bar{1}1}(\alpha, t) : e_{2k}e_{2m-1} = \alpha e_{2(k+m)-1+t}, \quad \alpha \in \{0, 1\}.$$

$$L_{\bar{1}2}(t) : \begin{cases} e_{2k}e_{2m-1} = -e_{2(k+m)-1+t} \\ e_{2k-1}e_{2m} = e_{2(k+m)-1+t}, \end{cases}$$

$$L_{\bar{1}3}(t) : e_{2k}e_{2m} = e_{2(k+m)+t}.$$

Proposition 4.9. *For any $k = 1, 2, 3$ we have $L_{\bar{0}k}(\theta, t) \cong L_{\bar{1}k}(\theta, t')$. Moreover the algebras $L_{\bar{0}1}, L_{\bar{0}2}, L_{\bar{0}3}$ are pairwise non isomorphic.*

Proof. Take the isomorphism $\varphi : L_{\overline{0}k}(\theta, t) \longrightarrow L_{\overline{1}k}(\theta, t')$ defined by $\varphi(e_i) = e'_{i+t-t'}$. Then the proof can be completed by verifying algebraic property of isomorphism. The statement that the algebras $L_{\overline{0}1}, L_{\overline{0}2}, L_{\overline{0}3}$ are pairwise non isomorphic follows from the property of the algebra to be Lie or commutative algebra, i.e., property to satisfy the different identities. \square

4.10. $3\mathbb{Z}$ -periodic Leibniz algebras. .

In the case where $G = \mathbb{Z}$, $\widehat{G} = 3\mathbb{Z}$ and $a \in \overline{k}, b \in \overline{i}, c \in \overline{j}, t \in \overline{s}$ we have:

$$\alpha_{ij}\alpha_{k,\overline{i+j+s}} = -\alpha_{ji}\alpha_{k,\overline{i+j+s}}.$$

Consider $\mathbb{Z}/3\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}\}$.

For $t \in \overline{0}$ from (3.4) we get the following

- 1) $a, b, c \in \overline{0} \Rightarrow \alpha_{00} = 0.$
- 2) $a, b \in \overline{0}, c \in \overline{1} \Rightarrow \alpha_{01}^2 = \alpha_{00}\alpha_{01} - \alpha_{01}\alpha_{10}.$
- 3) $a, c \in \overline{0}, b \in \overline{1} \Rightarrow \alpha_{10}\alpha_{01} = \alpha_{01}\alpha_{10} - \alpha_{00}\alpha_{01}.$
- 4) $b, c \in \overline{0}, a \in \overline{1} \Rightarrow \alpha_{00}\alpha_{10} = 0.$
- 5) $a, b, c \in \overline{1} \Rightarrow \alpha_{11}\alpha_{12} = 0.$
- 6) $a, b \in \overline{1}, c \in \overline{0} \Rightarrow \alpha_{10}\alpha_{11} = \alpha_{11}\alpha_{20} - \alpha_{10}\alpha_{11}.$
- 7) $a, c \in \overline{1}, b \in \overline{0} \Rightarrow \alpha_{01}\alpha_{11} = \alpha_{10}\alpha_{11} - \alpha_{11}\alpha_{20}.$
- 8) $b, c \in \overline{1}, a \in \overline{0} \Rightarrow \alpha_{11}\alpha_{02} = 0.$
- 1') $a, b \in \overline{0}, c \in \overline{2} \Rightarrow \alpha_{02}^2 = \alpha_{00}\alpha_{02} - \alpha_{02}\alpha_{20}.$
- 2') $a, c \in \overline{0}, b \in \overline{2} \Rightarrow \alpha_{20}\alpha_{02} = \alpha_{02}\alpha_{20} - \alpha_{00}\alpha_{02}.$
- 3') $b, c \in \overline{0}, a \in \overline{2} \Rightarrow \alpha_{00}\alpha_{20} = 0.$
- 4') $a, b, c \in \overline{2} \Rightarrow \alpha_{22}\alpha_{21} = 0.$
- 5') $a, b \in \overline{2}, c \in \overline{0} \Rightarrow \alpha_{20}\alpha_{22} = \alpha_{22}\alpha_{10} - \alpha_{20}\alpha_{22}.$
- 6') $a, c \in \overline{2}, b \in \overline{0} \Rightarrow \alpha_{02}\alpha_{22} = -\alpha_{20}\alpha_{22}.$
- 7') $b, c \in \overline{2}, a \in \overline{0} \Rightarrow \alpha_{22}\alpha_{01} = 0.$
- 1'') $a, b \in \overline{1}, c \in \overline{2} \Rightarrow \alpha_{12}\alpha_{10} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{01}.$
- 2'') $a, c \in \overline{1}, b \in \overline{2} \Rightarrow \alpha_{21}\alpha_{10} = -\alpha_{12}\alpha_{10}.$
- 3'') $b, c \in \overline{1}, a \in \overline{2} \Rightarrow \alpha_{11}\alpha_{22} = 0.$
- 4'') $a, b, c \in \overline{2} \Rightarrow \alpha_{22}\alpha_{21} = 0.$
- 5'') $a, b \in \overline{2}, c \in \overline{1} \Rightarrow \alpha_{21}\alpha_{20} = \alpha_{22}\alpha_{11} - \alpha_{21}\alpha_{02}.$
- 6'') $a, c \in \overline{2}, b \in \overline{1} \Rightarrow \alpha_{12}\alpha_{20} = -\alpha_{21}\alpha_{20}.$
- 7'') $b, c \in \overline{2}, a \in \overline{1} \Rightarrow \alpha_{22}\alpha_{11} = 0.$
- a) $a \in \overline{0}, b \in \overline{1}, c \in \overline{2} \Rightarrow \alpha_{12}\alpha_{00} = \alpha_{01}\alpha_{12} - \alpha_{02}\alpha_{21}.$
- a') $a \in \overline{0}, b \in \overline{2}, c \in \overline{1} \Rightarrow \alpha_{21}\alpha_{00} = -\alpha_{12}\alpha_{00}.$
- b) $a \in \overline{1}, b \in \overline{0}, c \in \overline{2} \Rightarrow \alpha_{02}\alpha_{12} = \alpha_{10}\alpha_{12} - \alpha_{12}\alpha_{00}.$
- b') $a \in \overline{1}, b \in \overline{2}, c \in \overline{0} \Rightarrow \alpha_{20}\alpha_{12} = -\alpha_{02}\alpha_{12}.$
- c) $a \in \overline{2}, b \in \overline{0}, c \in \overline{1} \Rightarrow \alpha_{01}\alpha_{21} = \alpha_{20}\alpha_{21} - \alpha_{21}\alpha_{00}.$
- c') $a \in \overline{2}, b \in \overline{1}, c \in \overline{0} \Rightarrow \alpha_{10}\alpha_{21} = -\alpha_{01}\alpha_{21}.$

Solving these equations we get the following eleven matrices:

$$\begin{aligned}
A_1 : & \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{10} & 0 & 0 \\ \alpha_{20} & 0 & 0 \end{pmatrix}; A_2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha_{21} \neq 0 & 0 \end{pmatrix}; A_3 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_{12} \neq 0 \\ 0 & \alpha_{21} & 0 \end{pmatrix}; \\
A_4 : & \begin{pmatrix} 0 & 0 & \alpha_{02} \neq 0 \\ \alpha_{10} & 0 & 0 \\ -\alpha_{02} & 0 & 0 \end{pmatrix}; A_5 : \begin{pmatrix} 0 & \alpha_{01} \neq 0 & 0 \\ -\alpha_{01} & 0 & 0 \\ \alpha_{20} & 0 & 0 \end{pmatrix}; \\
A_6 : & \begin{pmatrix} 0 & \alpha_{01} \neq 0 & \alpha_{02} \neq 0 \\ -\alpha_{01} & 0 & 0 \\ -\alpha_{02} & 0 & 0 \end{pmatrix}; A_7 : \begin{pmatrix} 0 & \alpha_{01} \neq 0 & -\alpha_{01} \\ -\alpha_{01} & 0 & \alpha_{12} \neq 0 \\ \alpha_{01} & -\alpha_{12} & 0 \end{pmatrix}; \\
A_8 : & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_{12} \\ 0 & 0 & \alpha_{22} \neq 0 \end{pmatrix}; A_9 : \begin{pmatrix} 0 & 0 & -\alpha_{20} \\ 2\alpha_{20} & 0 & 0 \\ \alpha_{20} \neq 0 & 0 & \alpha_{22} \neq 0 \end{pmatrix}; \\
A_{10} : & \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_{11} \neq 0 & 0 \\ 0 & \alpha_{21} & 0 \end{pmatrix}; A_{11} : \begin{pmatrix} 0 & \alpha_{01} \neq 0 & 0 \\ -\alpha_{01} & \alpha_{11} \neq 0 & 0 \\ -2\alpha_{01} & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Now by an appropriate scaling of basis we can take all non-zero parameters to be equal to 1. Thus we have proved the following:

Proposition 4.11. *For $t = 0$, there are the following eleven (infinite dimensional) $3\mathbb{Z}$ -periodic complex Leibniz algebras (all omitted products are to be understood as being equal to zero):*

$$\begin{aligned}
L_1(\alpha, \beta) : & \begin{cases} e_{3k+1}e_{3m} = \alpha e_{3(k+m)+1} \\ e_{3k+2}e_{3m} = \beta e_{3(k+m)+2} \end{cases} ; L_2 : e_{3k+2}e_{3m+1} = e_{3(k+m)+1}; \\
L_3(\beta) : & \begin{cases} e_{3k+1}e_{3m+2} = e_{3(k+m)+1} \\ e_{3k+2}e_{3m+1} = \beta e_{3(k+m)+1} \end{cases} ; L_4(\beta) : \begin{cases} e_{3k}e_{3m+2} = e_{3(k+m)+2} \\ e_{3k+2}e_{3m} = -e_{3(k+m)+2}, \\ e_{3k+1}e_{3m} = \beta e_{3(k+m)+1} \end{cases} ; \\
L_5(\beta) : & \begin{cases} e_{3k}e_{3m+1} = e_{3(k+m)+1} \\ e_{3k+1}e_{3m} = -e_{3(k+m)+1} \\ e_{3k+2}e_{3m} = \beta e_{3(k+m)+2} \end{cases} ; L_6(\beta) : \begin{cases} e_{3k}e_{3m+1} = e_{3(k+m)+1} \\ e_{3k+1}e_{3m} = -e_{3(k+m)+1} \\ e_{3k}e_{3m+2} = \beta e_{3(k+m)+2} \\ e_{3k+2}e_{3m} = -\beta e_{3(k+m)+2}, \end{cases} \quad \beta \neq 0; \\
L_7 : & \begin{cases} e_{3k}e_{3m+1} = e_{3(k+m)+1} \\ e_{3k+1}e_{3m} = -e_{3(k+m)+1}, \\ e_{3k}e_{3m+2} = -e_{3(k+m)+2} \\ e_{3k+2}e_{3m} = e_{3(k+m)+2} \\ e_{3k+1}e_{3m+2} = e_{3(k+m)+1} \\ e_{3k+2}e_{3m+1} = -e_{3(k+m)+1} \end{cases} ; L_8(\alpha) : \begin{cases} e_{3k+1}e_{3m+2} = \alpha e_{3(k+m)+1} \\ e_{3k+2}e_{3m+2} = e_{3(k+m)+1}, \end{cases} ;
\end{aligned}$$

$$L_9 : \begin{cases} e_{3k+2}e_{3m} = e_{3(k+m)+2} \\ e_{3k}e_{3m+2} = -e_{3(k+m)+2}, \\ e_{3k+1}e_{3m} = 2e_{3(k+m)+1} \\ e_{3k+2}e_{3m+2} = e_{3(k+m+1)+1} \end{cases} ; \quad L_{10}(\beta) : \begin{cases} e_{3k+1}e_{3m+1} = e_{3(k+m)+2} \\ e_{3k+2}e_{3m+1} = \beta e_{3(k+m+1)}, \end{cases} ;$$

$$L_{11} : \begin{cases} e_{3k}e_{3m+1} = e_{3(k+m)+1} \\ e_{3k+1}e_{3m} = -e_{3(k+m)+1}, \\ e_{3k+2}e_{3m} = -2e_{3(k+m)+2} \\ e_{3k+1}e_{3m+1} = e_{3(k+m)+2} \end{cases} \quad \text{where } k, m \in \mathbb{Z}.$$

As a corollary of Theorem 4.3 we have

Corollary 4.12. *A $3\mathbb{Z}$ -periodic Leibniz algebra L is right nilpotent if and only if $\alpha_{i0} = 0$, $\forall i = 0, 1, 2$, $\alpha_{01}\alpha_{11}\alpha_{21} = \alpha_{02}\alpha_{12}\alpha_{22} = 0$ and $\alpha_{01}\alpha_{12} = \alpha_{11}\alpha_{22} = \alpha_{21}\alpha_{02} = 0$.*

This corollary with Proposition 4.11 gives the following

Corollary 4.13. *The algebras L_2, L_3, L_8, L_{10} mentioned in Proposition 4.11 are nilpotent.*

4.14. $n\mathbb{Z}$ -periodic algebras. Denote $N_n = \{0, \dots, n-1\}$. The following proposition gives some properties of $n\mathbb{Z}$ -periodic complex Leibniz algebras:

Proposition 4.15. *Let L be an $n\mathbb{Z}$ -periodic Leibniz algebra. Then*

- 1) $L^2 = L$ if and only if for any $p \in N_n$ there exists $i \in N_n$ such that $\alpha_{i, \overline{p-i}} \neq 0$;
- 2) For any $n\mathbb{Z}$ -periodic Leibniz algebra L there exists an algebra L' such that $\alpha'_{00} = 0$ (i.e. $e_{nk}e_{nm} = 0$, $k, m \in \mathbb{Z}$) and $L \cong L'$;
- 3) $\{e_{ns+j} \mid s \in \mathbb{Z}\} \subseteq \text{Ann}_r(L)$ ($\{e_{ns+j} \mid s \in \mathbb{Z}\} \subseteq \text{Ann}_l(L)$) if and only if the j -th column (row) of the matrix A of structural constants of algebra L is zero;
- 4) The element e_{nk+i} , $i \neq 0$ is right (left) nilpotent if and only if in i -th column (row) of A there exists a zero;
- 5) The algebra L is a Lie algebra iff $A_n = (\alpha_{ij})_{i,j=0,\dots,n-1}$ is a skew-symmetric matrix.

Proof. 1) *Necessity.* If $L^2 = L$ then for any $e_c \in L$ there are $a, b \in \mathbb{Z}$ such that $\alpha_{\overline{a}, \overline{b}} \neq 0$ and

$$e_c = e_a e_b = \alpha_{\overline{a}, \overline{b}} e_{a+b}.$$

If $a \in \overline{i}$, $b \in \overline{j}$ and $c \in \overline{p}$ then from $c = a + b$ we get $j = \overline{p-i}$.

Sufficiency. Assume for any $p \in N_n$ there exists $i \in N_n$ such that $\alpha_{i, \overline{p-i}} \neq 0$. Define $a = i$ and $b = \overline{p-i}$ then

$$e_a e_b = \alpha_{i, \overline{p-i}} e_{a+b} = \alpha_{i, \overline{p-i}} e_c.$$

Hence $e_c \in L^2$ and $L^2 = L$.

2) Follows from Proposition 4.5 and Remark 4.6.

3) Straightforward.

4) This follows from the following equality and the property of the complete residue system

$$e_{nk+i}^{(n+1)r} = \alpha_{i,i} \alpha_{\overline{2i}, i} \dots \alpha_{\overline{ni}, i} e_{n(nk+k+i)+i} = \alpha_{0,i} \alpha_{1,i} \dots \alpha_{n-1,i} e_{n(nk+k+i)+i}.$$

5) It is easy to see that a Leibniz algebra $L(f, t)$ is a Lie algebra iff its matrix A of structural constants is skew-symmetric. The matrix A of a $n\mathbb{Z}$ -periodic algebra L is constructed by blocs A_n . From the following equalities it follows that A is skew-symmetric iff A_n is skew-symmetric:

$$\alpha_{nk+i, nl+j} = \alpha_{ij} = -\alpha_{ji} = -\alpha_{nl+j, nk+i}.$$

This completes the proof. \square

The following proposition gives a characterization of subalgebras.

Proposition 4.16. *Let $L(f, 0)$ be an $n\mathbb{Z}$ -periodic algebra with matrix $A_n = (\alpha_{ij})_{i,j=0,\dots,n-1}$ and let $s \in \{1, \dots, n-1\}$. If*

$$(\alpha_{is}, \alpha_{si}) \neq (0, 0), \quad \text{for all } i = 0, 1, \dots, n-1, \quad (4.6)$$

then $\langle e_{nk+s} : k \in \mathbb{Z} \rangle = L(f, 0)$.

Proof. From $e_{nk+s}e_{nm+s} = \alpha_{s,s}e_{n(k+m)+2s}$, by the condition $\alpha_{s,s} \neq 0$ we get $e_{n(k+m)+2s} \in \langle e_{nk+s} : k \in \mathbb{Z} \rangle$. Similarly, we get that $e_{nq+ps} \in \langle e_{nk+s} : k \in \mathbb{Z} \rangle$, for any $q \in \mathbb{Z}$ and $p \in \mathbb{N}$. Thus we get that the complete residue system $\{e_{nk+i}, i = 0, \dots, n-1; k \in \mathbb{Z}\}$, is contained in $\langle e_{nk+s} : k \in \mathbb{Z} \rangle = L(f, 0)$. \square

Remark 4.17. *Note that if $s \neq 0$ and $\langle e_{nk+s} : k \in \mathbb{Z} \rangle \subseteq I = \text{ideal}\langle xx : x \in L \rangle$ and the condition (4.6) is satisfied then $I = L$. Consequently, L is an Abelian algebra and $A_n = 0$, which is contradicts condition (4.6). Thus from $\langle e_{nk+s} : k \in \mathbb{Z} \rangle \subseteq I$ it follows that there exists $i \in \{0, \dots, n-1\}$ such that $\alpha_{is} = \alpha_{si} = 0$.*

Consider the algebras L_i , $i = 1, \dots, 11$ listed in Proposition 4.11. Using Proposition 3.2 we get the following:

Proposition 4.18. *1) Each of the algebras $L_1, L_2, L_3, L_4, L_5, L_6, L_8, L_{10}$ are solvable with index of solvability 2.*

2) The algebra L_7 is not solvable.

3) The algebras L_9, L_{11} are solvable with index of solvability 4.

Proof. 1) Solvability of $L_1 - L_6, L_8, L_{10}$ obvious.

2) We shall check that L_7 is not solvable. It is easy to see that $L_7^2 = L_7$, consequently $L_7^{[k]} = L_7$ for all $k \in \mathbb{N}$.

3) We have $L_{11}^{[2]} = \langle (e_i e_j)(e_k e_l); i, j, k, l \in \mathbb{Z} \rangle$. Considering all possible values of i, j, k, l we obtain $L_{11}^{[2]} \subset \langle e_{3i}, e_{3j+2}; i, j \in \mathbb{Z} \rangle$. Since $\alpha_{00} = \alpha_{22} = 0$, we get $L_{11}^{[3]} \subset \langle e_{3j+2}; j \in \mathbb{Z} \rangle$, now since $\alpha_{22} = 0$ we obtain $L_{11}^{[4]} = 0$. By a similar argument one can show that L_9 is also solvable with index 4. \square

Theorem 4.19. *The following infinite dimensional $3\mathbb{Z}$ -periodic Leibniz algebras (mentioned in Proposition 4.11) for fixed parameters α, β are pairwise non-isomorphic:*

$$L_1(\alpha, \beta), L_3(\beta), L_4(\beta), L_6(\beta), L_7, L_8(\alpha), L_9, L_{11}.$$

Proof. Since L_7 is not solvable, it is not isomorphic to the algebras L_i , $i \neq 7$. Since the algebras L_9, L_{11} are solvable with index 4, each of them is not isomorphic to each of the algebras $L_1 - L_6, L_8, L_{10}$. Moreover, since L_9 is not a Lie algebra, but L_{11} is a Lie algebra, they are not isomorphic. We have the following:

The algebra L_8 is isomorphic to L_{10} with the isomorphism:

$$e'_{3k} = e_{3(k+1)}, e'_{3k+1} = e_{3k+2}, e'_{3k+2} = e_{3(k+1)+1}, k \in \mathbb{Z}.$$

The algebra L_2 is isomorphic to $L_3(0)$ with the isomorphism:

$$e'_{3k} = e_{3k}, e'_{3k+1} = e_{3k+2}, e'_{3k+2} = e_{3k+1}, k \in \mathbb{Z}.$$

Moreover, $L_2 \subseteq L_3(\beta)$.

The algebra L_4 is isomorphic to L_5 with the isomorphism:

$$e'_{3k} = e_{3k}, e'_{3k+1} = e_{3k+2}, e'_{3k+2} = e_{3k+1}, k \in \mathbb{Z}.$$

$L_4(\alpha)$ is not isomorphic to the algebra $L_6(\beta)$. Indeed, $L_6(\beta)$ is a Lie algebra but $L_4(\alpha \neq 0)$ is not a Lie algebra. For $\alpha = 0$ we have $\{0\} \neq \text{Center}(L_4) = \{e_{3k+1} : k \in \mathbb{Z}\}$ but $\text{Center}(L_6) = \{0\}$. Similarly one can show that $L_1(\alpha, \beta)$ is not isomorphic to $L_6(\gamma)$.

$L_1(\alpha, \beta)$ is not isomorphic to $L_4(\gamma)$, because $L_4(0)$ is a Lie algebra but $L_1(\alpha, \beta)$ is not a Lie algebra. For $\gamma \neq 0$ we have $\text{Ann}_l(L_4) = \{0\}$ but $\text{Ann}_l(L_1) = \{e_{3m} : m \in \mathbb{Z}\}$.

$L_3(\beta)$ is not isomorphic to $L_8(\alpha)$. This follows from the following relations

$$L_3^2 = \text{Center}(L_3),$$

$$\text{Center}(L_8(\alpha \neq 0)) = \langle e_{3k} \rangle \subsetneq L_8^2(\alpha \neq 0),$$

$$\text{Center}(L_8(0)) = \langle e_{3k}, e_{3k+1} \rangle \supsetneq L_8^2(0) = \langle e_{3k+1} \rangle.$$

□

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